Differential Equations – applications

Example 1: Rate of change is a function of time

$$\left[\frac{dy}{dt} = f(t) \quad \rightarrow \quad y = \int f(t)\,dt\right]$$

Differential equation ↑   ↑ General Solution

Question: The full volume of an inflatable mattress is 240 000 cm^3 and the rate at which the volume is deflating is given by: $\frac{dV}{dt} = \frac{-20\ 000}{16t+1}$ cm^3/s . What volume of air will remain in the mattress after 10 seconds?

$$V = \int \frac{-20\ 000}{16t+1}\ dt = -20\ 000 \int \frac{1}{16t+1}\ dt = -20\ 000 \cdot \frac{1}{16} \cdot \ln|16t+1| + c$$

So $V = -1250 \ln(16t+1) + c$ [using the modulus sign only for the initial solution is sufficient]

Since the initial ($t = 0$) volume is 240 000 cm^3, the constant will be found using:

$$V(0) = -1250 \ln 1 + c = 240\ 000$$
$$c = 240\ 000 + 1250 \ln 1$$
$$c = 240\ 000$$

So the expression for the volume in the inflatable mattress is:

$$V = -1250 \ln(16t+1) + 240\ 000$$

Therefore the volume of air that remains in the mattress after 10 seconds is:

$V(10) = -1250 \ln(16(10)+1) + 240\ 000 = 233\ 648.2$ cm^3.

Example 2: Rate of change is directly proportional to the value itself

$$\left[\frac{dy}{dx} = ky \quad \rightarrow \quad y = Ae^{kx}\right]$$

Differential equation General Solution

The proof of the general solution to this differential equation:

$\frac{dy}{dx} = ky$
$dy = ky\ dx$
$\frac{1}{y}\ dy = k\ dx$
$\int \frac{1}{y}\ dy = \int k\ dx$
$\ln y = kx + c$
$y = e^{kx+c}$
$y = e^{kx} \cdot e^{c}$
$y = Ae^{kx}$ [e^{c} is a constant and can therefore be replaced with a letter, in this case A.]

Applications of this type of differential equation could include growth and decay, inflation, Newton's law of cooling etc.

Question: A microbiologist finds that for the one-celled *Paramecium* organism, the growth rate of a colony is given by the differential equation: $\frac{dP}{dt} = 0.12P$ where $P =$ the number of *Paramecia* in the colony, and $t =$ time in hours. Solve the equation to find the general solution. If the initial population of *Paramecia* is 200, how many *Paramecia* will there be in 24 hours?

$P = Ae^{0.12t}$ which is the general solution to this type of differential equation.

Since the initial ($t = 0$) population is 200, the constant A will be found using:
$P(0) = Ae^{0.12(0)} = Ae^{0} = A \cdot 1 = A = 200.$

So $A = 100$ and the general equation for the population of the colony is:
$P = 200e^{0.12t}$.

Therefore the number of *Paramecia* after 24 hours is:
$P(24) = 200e^{0.12(24)} = 200e^{2.88} \approx 3563$

Worksheet Ex 3.2C M7